Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

Monique Dauge, Patrick Dular, Laurent Krähenbühl, Victor Péron, Ronan Perrussel, Clair Poignard

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Victor Péron[♠] Ronan Perrussel[†] Clair Poignard[♣]

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JSA 2013,
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Rennes, August 26-30.

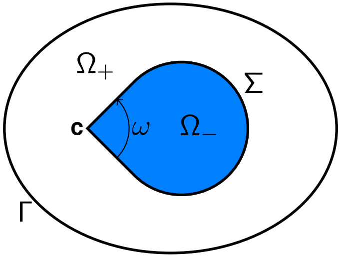
The Configuration

Ω_- : a conducting body ($\sigma > 0$)

Ω_+ : a dielectric medium

c : a corner

$\omega \in \big(0, 2\pi\big)$: the angle of the corner



We describe the Magnetic potential in the vicinity of the corner **c** .

The Magnetic Potential

The eddy-current problem

The magnetic vector potential $\mathcal{A}$ satisfies

$$\begin{cases} -\Delta\mathcal{A}^- + 4\mathrm{i}\zeta^2\mathcal{A}^- = 0 \text{ in } \Omega_-, \\ -\Delta\mathcal{A}^+ = \mu_0 J \text{ in } \Omega_+, \\ \mathcal{A}^+ = 0 \text{ on } \Gamma, \end{cases} \quad \begin{aligned} [\mathcal{A}]_\Sigma &= 0, \text{ on } \Sigma, \\ [\partial_n\mathcal{A}]_\Sigma &= 0, \text{ on } \Sigma. \end{aligned} \tag{1}$$

Here $\zeta^2 = \kappa\mu_0\sigma/4 > 0$,

J : a smooth data, vanishing near the corner **c**.

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Here $\zeta^2 = \kappa\mu_0\sigma/4 > 0$,

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Proposition

There exists a unique solution $\mathcal{A}$ in $H^1_0(\Omega)$ to problem (1). Moreover, $\mathcal{A}$ belongs to $H^{\frac{5}{2}-\varepsilon}(\Omega)$ for any $\varepsilon > 0$. In particular, $\mathcal{A}$ belongs to $\mathcal{C}^1(\overline{\Omega})$.

$\mathcal{A}$ possesses a corner asymptotic expansion near **c**.

Corner Asymptotics

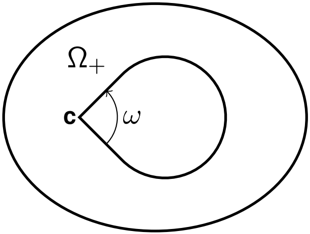
- To generalize the Taylor Expansion
- Corner Asymptotics involve
  1. The singular functions (primal and dual) :
     belong to the kernel of the considered operator in $\mathbb{R}^2$.
  2. The singular coefficients :
     its calculation requires the knowledge of dual singular functions.

Aim : To explicit the Corner Asymptotics of $\mathcal{A}$ near the corner $\mathbf{c}$.

Motivation : The eddy-current phenomenon

- The limit problem in large frequency/high conductivity :

$$\begin{cases} -\Delta \mathcal{A}_0^+ = \mu_0 J \text{ in } \Omega_+, \\ \mathcal{A}_0^+ = 0 \text{ on } \partial\Omega_+. \end{cases} \tag{2}$$



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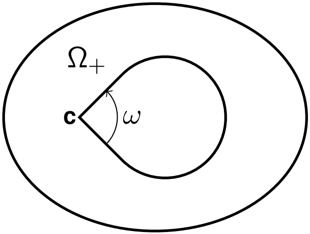

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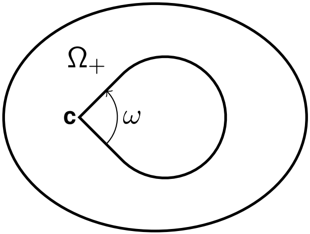

- If Ω_- has a convex corner, i.e. $\omega \in (0, \pi)$, Problem (2) has non $\mathcal{C}^1$ singularities
- Problem (1) has $\mathcal{C}^1$ singularities

Motivation : Multi-scale analysis for eddy-currents

BURET *et al* (IEEE Trans. on Mag. '12)
Eddy currents and corner singularities

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- When $\delta = \sqrt{\frac{2}{\kappa\mu_0\sigma}} \ll 1$, there holds

$$\mathcal{A}_\delta \approx \mathcal{A}_0 + a\,\delta^\alpha\, V\Big(\frac{\cdot}{\delta}\Big) + R_\delta\ .$$

Here V is a *profile* defined in $\mathbb{R}^2$ such that

$$-\Delta V + 2iV\,\mathbb{1}_{\Omega_-} = 0 \quad \text{near } \mathbf{c},$$

and $\alpha = \frac{\pi}{2\pi-\omega}$.

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$$-\Delta V + 2iV\,\mathbb{1}_{\Omega_-} = 0 \quad \text{near } \mathbf{c},$$

and $\alpha = \frac{\pi}{2\pi-\omega}$.

$\Longrightarrow$ The knowledge of the *singularities* of $-\Delta + 4i\zeta^2\,\mathbb{1}_{\Omega_-}$ is "crucial".

Main difficulty

The considered operator is

$$-\Delta + i\kappa\mu_0\sigma\,\mathbb{1}_{\Omega_-} = \begin{cases} -\Delta + i\kappa\mu_0\sigma & \text{in } \Omega_-, \\ -\Delta & \text{in } \Omega_+. \end{cases}$$

The *singularities* are generated by the term $i\kappa\mu_0\sigma\,\mathbb{1}_{\Omega_-}$.

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$\Longrightarrow$ the derivation of the Corner Asymptotics is not obvious.

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M. DAUGE *et al* (To appear in MMAS - INRIA RR 8204)
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- $\mathcal{A}$ possesses a corner asymptotic expansion

$$\mathcal{A}(r,\theta) \underset{r\to 0}{\sim} \Lambda^{0,0}\mathfrak{S}^{0,0}(r,\theta) + \sum_{k\geqslant 1}\sum_{p\in\{0,1\}} \Lambda^{k,p}\mathfrak{S}^{k,p}(r,\theta),$$

(r,θ) : polar coordinates centered at $\mathbf{c}$
$\mathfrak{S}^{k,p}$: primal singular functions
$\Lambda^{k,p}$: singular coefficients

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- We provide a constructive procedure to determine the primal and dual singularities

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(r,θ) : polar coordinates centered at **c**
$\mathfrak{S}^{k,p}$: primal singular functions
$\Lambda^{k,p}$: singular coefficients

- We provide a constructive procedure to determine the primal and dual singularities
- We generalize the method of moments and we introduce the method of quasi-dual functions to determine $\Lambda^{k,p}$

Outline

1. **The case $\zeta = 0$:**

   We introduce the method of moments and the method of dual functions.

2. **The case $\zeta \neq 0$:**

   We provide a constructive procedure to determine the singularities.

   We introduce the method of quasi-dual singular functions.

3. **Numerical simulations**

Laplace Operator ($\zeta = 0$)

- We consider the solution $\mathcal{A}$ to

$$\begin{cases} -\Delta\mathcal{A} = \mu_0 J \text{ in } \Omega \\ \mathcal{A} = 0 \text{ on } \Gamma \end{cases}$$

- $\mathcal{A}$ admits the Taylor expansion at **c** :

$$\mathcal{A}(r,\theta) \underset{r\to 0}{\sim} \Lambda^{0,0} + \sum_{k\geqslant 1} \sum_{p\in\{0,1\}} \Lambda^{k,p} \underbrace{r^k \cos(k\theta - p\pi/2)}_{=\mathfrak{s}^{k,p}(r,\theta)}$$

$\mathfrak{s}^{k,p}$: harmonic polynomials

Methods to extract the coefficients $\Lambda^{k,p}$

We use dual harmonic functions :

$$\mathfrak{k}^{k,p}(r,\theta) = \begin{cases} -\dfrac{1}{2\pi}\,\log r, & \text{if } k=0,\ p=0, \\ \dfrac{1}{2k\pi}\,r^{-k}\cos\big(k\theta - p\pi/2\big), & \text{if } k \geqslant 1,\ p=0,1. \end{cases}$$

- The method of moments
- The dual function method

The Method of Moments

For $R > 0$, we introduce the form $\mathcal{M}_R$:

$$\mathcal{M}_R(K, A) = \frac{1}{R} \int_{r=R} K\, A\, R \mathrm{d}\theta.$$

Assume J has a support outside the ball $\mathcal{B}(\mathbf{c}, R)$.

Proposition

Let $\mathcal{A}$ be the solution to the Laplace equation. Then

$$\mathcal{M}_R(1, \mathcal{A}) = 2\pi \Lambda^{0,0} \quad \text{and} \quad \mathcal{M}_R(\mathfrak{r}^{k,p}, \mathcal{A}) = \frac{1}{2k} \Lambda^{k,p}, \;\; k \geqslant 1, \;\; p = 0, 1.$$

The Dual Function Method

V.G. MAZ'YA, B.A. PLAMENEVSKII (Amer. Math. Soc. Trans. (2) '84)
On the coefficients in the asymptotic of solutions of the elliptic boundary problem in domains with conical points

- For $R > 0$, let us introduce the bilinear form

$$\mathcal{J}_R(K, A) = \int_{r=R} \left(K\partial_r A - \partial_r K\, A\right)\, R\mathrm{d}\theta.$$

Proposition

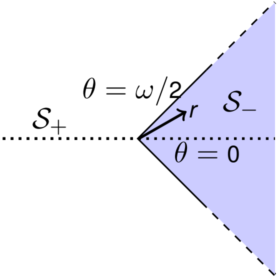
Let $\mathcal{A}$ be the solution to the Laplace equation. Then

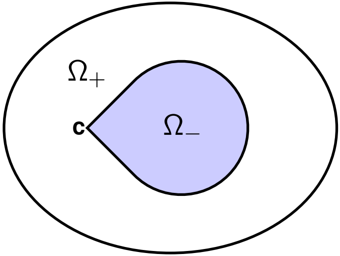
$$\mathcal{J}_R(\mathfrak{k}^{0,0}, \mathcal{A}) = \Lambda^{0,0} \quad \text{and} \quad \mathcal{J}_R(\mathfrak{k}^{k,p}, \mathcal{A}) = \Lambda^{k,p}, \;\; k \geqslant 1,\; p = 0, 1.$$

The *singularities*

The *singularities* $\mathfrak{U}$ of $-\Delta + 4\mathrm{i}\zeta^2\,\mathbb{1}_{\mathcal{S}_-}$:

$$\begin{cases} -\Delta\mathfrak{U} + 4\mathrm{i}\zeta^2\mathfrak{U} = 0 & \text{in } \mathcal{S}_-, \\ -\Delta\mathfrak{U} = 0 & \text{in } \mathcal{S}_+. \end{cases}$$





Description of the *Singularities*

- V.A. KONDRATEV (Trudy Moskov. Mat. Obšč. '67)
  Boundary value problems for elliptic equations in domains with conical or angular points

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- $$\mathfrak{U} = \underbrace{\mathfrak{u}_0}_{\textit{leading part}} + \sum_{j \geqslant 1} (\mathrm{i}\zeta^2)^j \underbrace{\mathfrak{u}_j}_{\textit{shadow}} ,$$

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- $$\mathfrak{U} = \underbrace{\mathfrak{u}_0}_{\text{leading part}} + \sum_{j\geqslant 1} (\mathrm{i}\zeta^2)^j \underbrace{\mathfrak{u}_j}_{\text{shadow}},$$

- We solve

$$\Delta \mathfrak{u}_0 = 0, \quad \Delta \mathfrak{u}_1 = 4\mathfrak{u}_0 \, \mathbb{1}_{\mathcal{S}_-}, \quad \dots, \quad \Delta \mathfrak{u}_j = 4\mathfrak{u}_{j-1} \, \mathbb{1}_{\mathcal{S}_-},$$

$$\mathfrak{u}_j \in \mathsf{S}^\lambda = \mathrm{Span}\left\{ r^\lambda \log^q r \, \Phi(\theta), \quad q \in \mathbb{N}, \;\; \Phi \in \mathcal{C}^1(\mathbb{T}), \; \Phi^\pm \in \mathcal{C}^\infty(\overline{\mathbb{T}}_\pm) \right\}$$

Here $\lambda \in \mathbb{Z}$,
$\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, $\mathbb{T}_- = (-\omega/2, \omega/2)$ and $\mathbb{T}_+ = \mathbb{T} \setminus \overline{\mathbb{T}_-}$.

Existence of the shadows

Lemma

Let $\lambda \in \mathbb{Z}$ and $\mathfrak{f} \in \mathsf{T}^{\lambda-2}$. Then, there exists $\mathfrak{u} \in \mathsf{S}^{\lambda}$ such that $\Delta\mathfrak{u} = \mathfrak{f}$. Moreover

- ***(i)*** *If $\lambda \in \mathbb{Z} \setminus \{0\}$, $\deg \mathfrak{u} \leqslant \deg \mathfrak{f} + 1$,*
- ***(ii)*** *If $\lambda = 0$, $\deg \mathfrak{u} \leqslant \deg \mathfrak{f} + 2$.*

Here,

$$\mathsf{T}^{\lambda} = \mathrm{Span}\left\{ r^{\lambda} \log^{q} r\, \Psi(\theta), \quad q \in \mathbb{N}, \;\; \Psi \in L^2(\mathbb{T}), \; \Psi^{\pm} \in \mathcal{C}^{\infty}(\overline{\mathbb{T}}_{\pm}) \right\}.$$

The degree of $\mathfrak{g}$ is its degree as polynomial of $\log r$.

Proof of Lemma

M. Dauge *Elliptic Boundary Value Problems in Corner Domains – Smoothness and Asymptotics of Solutions, Lecture Notes in Mathematics, Vol. 1341, Berlin 1988*

We introduce the *Mellin symbol* $\mathscr{M}$ associated with Δ

$$\mathscr{M}(\mu) = (\mu^2 + \partial_\theta^2),\ \theta \in \mathbb{T}.$$

Setting

$$\mathfrak{f} = \sum_{q=0}^{\deg \mathfrak{f}} r^{\lambda-2} \log^q r\, \Psi_q \in \mathsf{T}^{\lambda-2},$$

and

$$\mathfrak{u} = \operatorname*{Res}_{\mu=\lambda} \left\{ r^\mu \mathscr{M}(\mu)^{-1} \sum_{q=0}^{\deg \mathfrak{f}} \frac{q!\, \Psi_q}{(\mu-\lambda)^{q+1}} \right\} \in \mathsf{S}^\lambda,$$

we check that $\Delta \mathfrak{u} = \mathfrak{f}$.

Primal and Dual Singularities

- *Primal singular functions* $\mathfrak{S}$ belong to H^1 in any bounded neighborhood $\mathcal{B}$ of $\mathbf{c}$.
- *Dual singular functions* $\mathfrak{K}$ do not belong to H^1 in such a neighborhood $\mathcal{B}$.

The functions $\mathfrak{K}$ are needed for the determination of the coefficients Λ involved in the asymptotics.

1. A basis for Primal Singularities

For $(k,p) = (0,0)$ and for $(k,p) \in \mathbb{N}^* \times \{0,1\}$,

$$\mathfrak{S}^{k,p} = \underbrace{\mathfrak{s}^{k,p}}_{\textit{leading part}} + \sum_{j \geqslant 1} (\mathrm{i}\zeta^2)^j \underbrace{\mathfrak{s}_j^{k,p}}_{\textit{shadow}} .$$

Lemma

For any $j \geqslant 1$, there exists $\mathfrak{s}_j^{k,p} \in \mathsf{S}^{k+2j}$, with $\deg \mathfrak{s}_j^{k,p} \leqslant j$, satisfying

$$\Delta \mathfrak{s}_j^{k,p} = 4 \mathfrak{s}_{j-1}^{k,p} \, \mathbb{1}_{\mathcal{S}_-} .$$

Application : $\mathcal{A}(r,\theta) \underset{r \to 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r,\theta) + \sum_{k \geqslant 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} \mathfrak{S}^{k,p}(r,\theta)$

For $(k,p) = (0,0)$ and for $(k,p) \in \mathbb{N}^* \times \{0,1\}$,

$$\mathfrak{K}^{k,p} = \underbrace{\mathfrak{k}^{k,p}}_{\textit{leading part}} + \sum_{j \geqslant 1} (\mathrm{i}\zeta^2)^j \underbrace{\mathfrak{k}_j^{k,p}}_{\textit{shadow}} .$$

Lemma

For any $j \geqslant 1$, there exists $\mathfrak{k}_j^{k,p} \in \mathsf{S}^{-k+2j}$, with $\deg \mathfrak{k}_j^{k,p} \leqslant j+1$, satisfying

$$\Delta \mathfrak{k}_j^{k,p} = 4 \mathfrak{k}_{j-1}^{k,p} \, \mathbb{1}_{\mathcal{S}_-} .$$

3. The Quasi-Dual Function Method

- M. Costabel *et al* (SIAM '04)
  A quasidual function method for extracting edge stress intensity functions
- S. Shannon *et al* (Preprint '12)
  Extracting generalized edge flux intensity functions by the quasidual function method along circular 3-D edges
- We use quasi-dual functions $\mathfrak{K}_m^{k,p}$:

$$\mathfrak{K}_m^{k,p} = \mathfrak{k}^{k,p} + \sum_{j=1}^{m} (\mathrm{i}\zeta^2)^j \underbrace{\mathfrak{k}_j^{k,p}}_{\textit{shadow}}$$

Theorem

Let $\mathcal{A}$ be the solution to problem (1)*. Let $k \in \mathbb{N}$ and $p \in \{0, 1\}$ ($p = 0$ if $k = 0$). Then for all m such that $2m + 2 > k$,*

$$\mathcal{J}_R(\mathfrak{K}_m^{k,p}, \mathcal{A}) \underset{R\to 0}{=} \Lambda^{k,p} + \sum_{\ell=1}^{[k/2]} \mathcal{J}^{k,p\,;\,k-2\ell,p} \Lambda^{k-2\ell,p} + \mathcal{O}\big(R^{-k} R_0^{2m+2} \log R\big)\,,$$

where $\boxed{R_0 = \zeta R \left(1 + \sqrt{|\log R|}\right)}$.

The Quasi-Dual Function Method

Examples

- For $k = 0$

$$\Lambda^{0,0} \underset{R\to 0}{=} \mathcal{J}_R(\mathfrak{K}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R)$$

- For $k = 1$

$$\Lambda^{1,0} \underset{R\to 0}{=} \mathcal{J}_R(\mathfrak{K}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1} R_0^{2m+2})$$

- For $k = 2$, we need $m \geqslant 1$

$$\Lambda^{2,0} \underset{R\to 0}{=} \mathcal{J}_R(\mathfrak{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0\,;\,0,0}\Lambda^{0,0} + \mathcal{O}(R^{-2} R_0^4 \log R)$$

We use quasi-primal singularities $\mathfrak{S}_m^{k,p}$:

$$\mathfrak{S}_m^{k,p} = \mathfrak{s}^{k,p} + \sum_{j=1}^{m} (\mathrm{i}\zeta^2)^j \underbrace{\mathfrak{s}_j^{k,p}}_{shadow}$$

For $\varepsilon \in \big(0, R\big)$, we evaluate

$$\mathcal{J}_R\big(\mathfrak{K}_m^k, \mathfrak{S}_{m'}^{k'}\big) - \mathcal{J}_\varepsilon\big(\mathfrak{K}_m^k, \mathfrak{S}_{m'}^{k'}\big) \underset{R\to 0}{=} \mathcal{O}\big(R^{-k+k'} R_0^{2m+2} \log R\big)$$

with any chosen k', and $m' \geqslant m$.

4. Calculation of singularities

- M. COSTABEL, M. DAUGE (Math. Nachr. '93)
  Construction of corner singularities for Agmon-Douglis-Nirenberg elliptic systems.

  We use appropriate complex variables $z_{\pm}$ instead of the polar coordinates :
  $z_{-} = z$ (when $z \in \mathcal{S}_{-}$) and $z_{+} = -z$ (when $z \in \mathcal{S}_{+}$)
- We use an Ansatz involving only integer powers of

$$z_{\pm}\ ,\ \overline{z}_{\pm}\ ,\ \log z_{\pm}\ ,\ \text{and}\ \ \log \overline{z}_{\pm}\ .$$

The first shadow of primal singularities

Example : expression of $\mathfrak{s}_1^{0,0}$

$$\mathfrak{s}_1^{0,0}(r,\theta) = \frac{\sin\omega}{\pi}\, r^2\Big(\log r\ \cos 2\theta - \theta\sin 2\theta\Big) + r^2\Big(1 - \frac{\cos\omega\ \cos 2\theta}{2}\Big) \quad \text{in } \mathcal{S}_-$$

$$\mathfrak{s}_1^{0,0}(r,\theta) = \frac{\sin\omega}{\pi}\, r^2\Big(\log r\ \cos 2\theta - \theta_+\sin 2\theta\Big) + r^2\ \frac{\cos\omega\ \cos 2\theta}{2} \quad \text{in } \mathcal{S}_+$$

with $\theta_+ = \theta - \pi\,\mathrm{sgn}\,\theta$

The first shadow of primal singularities

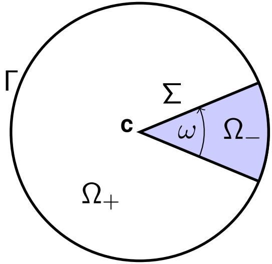
Example : Expression of $\mathfrak{s}_1^{1,0}$

$$\mathfrak{s}_1^{1,0}(r,\theta) = \frac{2\sin\omega + \sin 2\omega}{6\pi}\; r^3\Big(\log r\;\cos 3\theta - \theta\sin 3\theta\Big) + r^3\Big(\frac{\cos\theta}{2} - \frac{\cos\omega\;\cos 3\theta}{3}\Big), \quad \text{in} \quad \mathcal{S}_-$$

$$\mathfrak{s}_1^{1,0}(r,\theta) = \frac{2\sin\omega + \sin 2\omega}{6\pi}\; r^3\Big(\log r\;\cos 3\theta - \theta_+\sin 3\theta\Big) + r^3\,\frac{\cos 2\omega\;\cos 3\theta}{6}, \quad \text{in} \quad \mathcal{S}_+$$

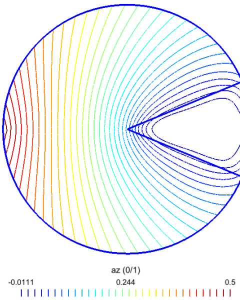
Framework

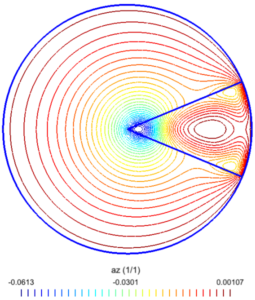
- Ω : disk of radius 50 mm
- Conducting sector : $\omega = \pi/4$
- $\zeta = 1/(5\sqrt{2})$ mm^{-1}
- Source : $\mathcal{A}^+ = \frac{|\theta|}{2\pi}$ on $\partial\Omega$



Finite Element Solution

We plot the real part and the imaginary part of the FE solution.



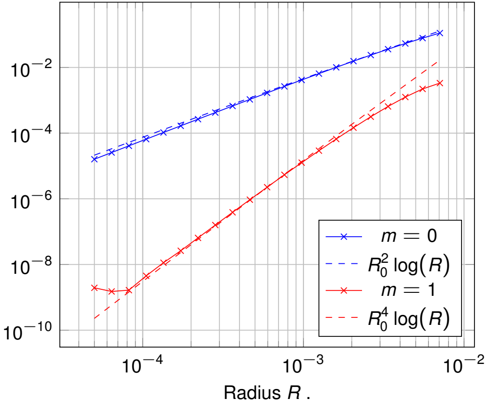


Accuracy for the computation of $\Lambda^{0,0}$ as a function of R

- There holds

$$\Lambda^{0,0} \underset{R\to 0}{=} \mathcal{J}_R(\mathfrak{K}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R) \, .$$

- We plot $|\mathcal{J}_R(\mathfrak{K}_m^{0,0}, \mathcal{A}) - \mathcal{A}|_{\mathbf{c}}|/|\mathcal{A}|_{\mathbf{c}}|$ as a function of R ($m = 0, 1$).

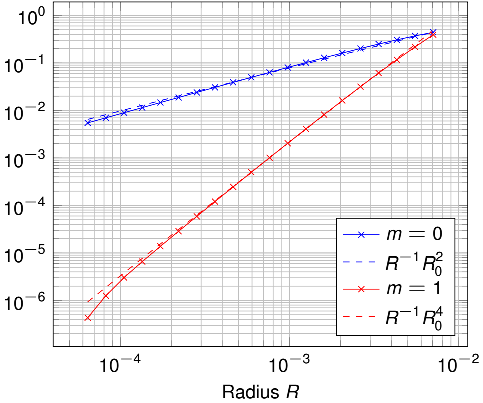


Accuracy for the computation of $\Lambda^{1,0}$ as a function of R

- Recall that

$$\Lambda^{1,0} \underset{R\to 0}{=} \mathcal{J}_R(\mathfrak{K}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1} R_0^{2m+2}) \, .$$

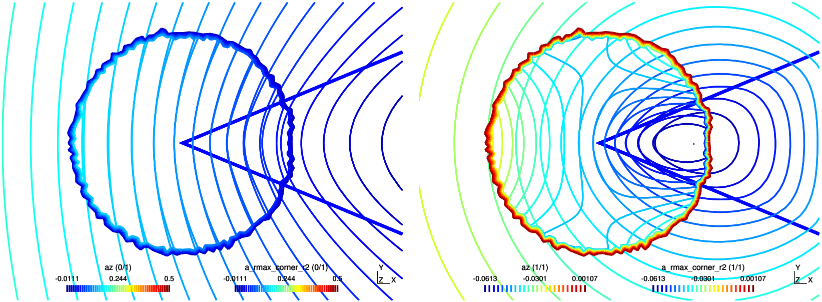
- We plot $|\mathcal{J}_R(\mathfrak{K}_m^{1,0}, \mathcal{A}) - \mathcal{J}_{R_s}(\mathfrak{K}_1^{1,0}, \mathcal{A})|/|\mathcal{J}_{R_s}(\mathfrak{K}_1^{1,0}, \mathcal{A})|$.



Comparison of the FE solution and of the local expansion

In the expansion, we collect the terms which behave as constant, r, r^2 and $r^2 \log r$.

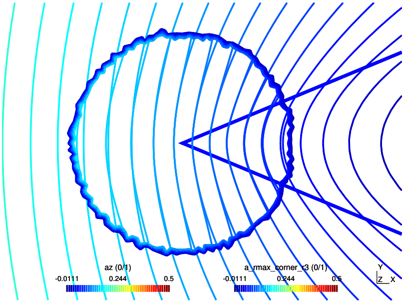
$$\mathcal{A} \underset{r\to 0}{\sim} \mathcal{J}_{R_\mathrm{s}}(\mathfrak{K}_1^{0,0},\mathcal{A})\left(1+\mathrm{i}\zeta^2\mathfrak{s}_1^{0,0}\right)+\mathcal{J}_{R_\mathrm{s}}(\mathfrak{K}_1^{1,0},\mathcal{A})\mathfrak{s}^{1,0}$$
$$+\left(\mathcal{J}_{R_\mathrm{s}}(\mathfrak{K}_1^{2,0},\mathcal{A})-\mathcal{J}^{2,0;0,0}\mathcal{J}_{R_\mathrm{s}}(\mathfrak{K}_1^{0,0},\mathcal{A})\right)\mathfrak{s}^{2,0}$$

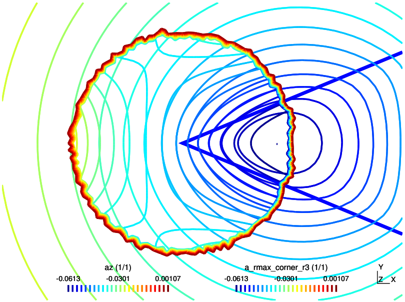


Comparison of the FE solution and of the local expansion

Then we add the terms which behave as r^3 and $r^3 \log r$.

$$\mathcal{A} \underset{r\to 0}{\sim} \mathcal{J}_{R_s}(\mathfrak{K}_1^{0,0},\mathcal{A})\left(1+\mathrm{i}\zeta^2\mathfrak{s}_1^{0,0}\right)+\mathcal{J}_{R_s}(\mathfrak{K}_1^{1,0},\mathcal{A})\left(\mathfrak{s}^{1,0}+\mathrm{i}\zeta^2\mathfrak{s}_1^{1,0}\right)+$$
$$\left(\mathcal{J}_{R_s}(\mathfrak{K}_1^{2,0},\mathcal{A})-\mathcal{J}^{2,0;0,0}\mathcal{J}_{R_s}(\mathfrak{K}_1^{0,0},\mathcal{A})\right)\mathfrak{s}^{2,0}+$$
$$\left(\mathcal{J}_{R_s}(\mathfrak{K}_1^{3,0},\mathcal{A})-\mathcal{J}^{3,0;1,0}\mathcal{J}_{R_s}(\mathfrak{K}_1^{1,0},\mathcal{A})\right)\mathfrak{s}^{3,0}$$





Thank you for your attention